

Non-colocated Time-Reversal MUSIC: High-SNR Distribution of Null Spectrum

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Abstract—We derive the asymptotic distribution of the null spectrum of the well-known Multiple Signal Classification (MUSIC) in its computational Time-Reversal (TR) form. The result pertains to a single-frequency non-colocated multistatic scenario and several TR-MUSIC variants are here investigated. The analysis builds upon the 1st-order perturbation of the singular value decomposition and allows a simple characterization of null-spectrum moments (up to the 2nd order). This enables a comparison in terms of spectrums stability. Finally, a numerical analysis is provided to confirm the theoretical findings.

Index Terms—Time-Reversal (TR), Radar imaging, Null-spectrum, Resolution, TR-MUSIC.

I. INTRODUCTION

TIME-REVERSAL (TR) refers to all those methods which exploit the invariance of the wave equation (in lossless and stationary media) by re-transmitting a time-reversed version of the scattered (or radiated) field measured by an array to focus on a scattering object (or radiating source), by physical [1] or synthetic [2] means. In the latter case (*computational TR*), it consists in *numerically* back-propagating the field data by using a known Green's function, representative of the propagation medium. Since the employed Green function depends on the object (or source) position, an image is formed by varying the probed location (this procedure is referred to as “imaging”). Computational TR has been successfully applied in different contexts such as subsurface prospecting [3], through-the-wall [4] and microwave imaging [5].

The key entity in TR-imaging is the Multistatic Data Matrix (MDM), whose entries are the scattered field due to each Transmit-Receive (Tx-Rx) pair. Two popular methods for TR-imaging are the decomposition of TR operator (DORT) [6] and the TR Multiple Signal Classification (TR-MUSIC) [7]. DORT imaging exploits the MDM spectrum by back-propagating each eigenvector of the so-called *signal subspace*, thus allowing to selectively focus on each (well-resolved)

scatterer. On the other hand, TR-MUSIC imaging is based on a complementary point of view and relies on the *noise subspace* (viz. orthogonal-subspace¹), leading to satisfactory performance as long as the data space dimension exceeds the signal subspace dimension and sufficiently high Signal-to-Noise-Ratio (SNR) is present. TR-MUSIC was first introduced for a Born Approximated (BA) linear scattering model [7] and, later, successfully applied to the Foldy-Lax (FL) non-linear model [8]. Also, it became popular mainly due to: (a) algorithmic efficiency; (b) no need for approximate scattering models; and (c) finer resolution than the diffraction limits (especially in scenarios with few scatterers). Recently, TR-MUSIC has been expanded to extended scatterers in [9].

Though a vast literature on performance analysis of MUSIC [10] for Direction-Of-Arrival (DOA) estimation exists (see [11], [12] for resolution studies and [13]–[16] for asymptotic Mean Squared Error (MSE) derivation, with more advanced studies presented in [17]–[19]), such results cannot be directly applied to TR-MUSIC. Indeed, in TR framework scatterers/sources are generally assumed deterministic and more importantly *a single snapshot is used*, whereas MUSIC results for DOA refer to a different asymptotic condition (i.e. a large number of snapshots). Also, to our knowledge, *no corresponding theoretical results have been proposed in the literature for TR-MUSIC*, except for [20], [21], providing the asymptotic (high-SNR) localization MSE for point-like scatterers. Yet, a few works have tackled achievable theoretical performance both for BA and FL models via the Cramér-Rao lower-bound [22].

In this letter we provide a null-spectrum² analysis of TR-MUSIC for point-like scatterers, via a 1st-order perturbation of Singular Value Decomposition (SVD) [24], thus having asymptotic validity (i.e. meaning a high SNR regime). The present results are based on a homogeneous background assumption and neglecting mutual coupling, as well as polarization or antenna pattern effects. Here we build upon [25] (tackling the simpler colocated case) and consider a *general* non-colocated multistatic setup with BA/FL models where several TR-MUSIC variants, proposed in the literature, are here investigated. The obtained results complement those found in DOA literature [23] and allow to obtain both the mean and the variance of each null-spectrum, as well as to draw-out its pdf. Also, they highlight performance dependence of null-spectrum on the scatterers/arrays configurations and compare TR-MUSIC variants in terms of spectrum stability.

¹Such term underlines that it is orthogonal to the signal subspace.

²We underline that the MUSIC imaging function is commonly referred to as “pseudo-spectrum” in DOA literature. Though less used, in this paper we will instead adopt to the term “null-spectrum” employed in [23], as the latter work represents the closest counterpart in DOA estimation to the present study.

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⁰*Notation* - Lower-case (resp. Upper-case) bold letters denote column vectors (resp. matrices), with a_n (resp. $a_{n,m}$) being the n th (resp. the (n,m) th) element of \mathbf{a} (resp. \mathbf{A}); $\mathbb{E}\{\cdot\}$, $\text{var}\{\cdot\}$, $(\cdot)^T$, $(\cdot)^\dagger$, $(\cdot)^*$, $\text{Tr}[\cdot]$, $\text{vec}(\cdot)$, $(\cdot)^{-}$, $\Re(\cdot)$, $\delta(\cdot)$, $\|\cdot\|_F$ and $\|\cdot\|$ denote expectation, variance, transpose, Hermitian, conjugate, matrix trace, vectorization, pseudo-inverse, real part, Kronecker delta, Frobenius and ℓ_2 norm operators, respectively; j denotes the imaginary unit; $\mathbf{0}_{N \times M}$ (resp. \mathbf{I}_N) denotes the $N \times M$ null (resp. identity) matrix; $\mathbf{0}_N$ (resp. $\mathbf{1}_N$) denotes the null (resp. ones) column vector of length N ; $\text{diag}(\mathbf{a})$ denotes the diagonal matrix obtained from the vector \mathbf{a} ; $\mathbf{x}_{1:M} \triangleq [\mathbf{x}_1^T \cdots \mathbf{x}_M^T]^T$ denotes the vector concatenation; $\mathcal{N}_{\mathbb{C}}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes a proper complex Gaussian pdf with mean vector $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$; χ_N^2 denotes a complex chi-square distribution with N (complex) Degrees of Freedom (DOFs); finally the symbol \sim means “distributed as”.

We recall that stability property is important for TR-MUSIC, and has been investigated by numerical means [26], [27] or using compressed-sensing based approaches [28]. Finally, a few numerical examples, for a 2-D geometry with scalar scattering, are presented to confirm our findings.

The letter is organized as follows: Sec. II describes the system model and reviews classic results on SVD perturbation analysis. Sec. III presents the theoretical characterization of TR-MUSIC null-spectrum, whereas its validation is shown in Sec. IV via simulations. Finally, conclusions are in Sec. V.

II. SYSTEM MODEL

We consider localization of M point-like scatterers³ at unknown positions $\{\mathbf{x}_k\}_{k=1}^M$ in \mathbb{R}^p with unknown scattering potentials $\{\tau_k\}_{k=1}^M$ in \mathbb{C} . The Tx (resp. Rx) array consists of N_T (resp. N_R) isotropic point elements (resp. receivers) located at $\{\tilde{\mathbf{r}}_i\}_{i=1}^{N_T}$ in \mathbb{R}^p (resp. $\{\bar{\mathbf{r}}_j\}_{j=1}^{N_R}$ in \mathbb{R}^p). The illuminators first send signals to the probed scenario (in a known homogeneous background with wavenumber κ) and the transducer array records the received signals. The (single-frequency) measurement model is then [30]:

$$\begin{aligned} \mathbf{K}_n &= \mathbf{K}(\mathbf{x}_{1:M}, \boldsymbol{\tau}) + \mathbf{W} \\ &= \mathbf{G}_t(\mathbf{x}_{1:M}) \mathbf{M}(\mathbf{x}_{1:M}, \boldsymbol{\tau}) \mathbf{G}_r(\mathbf{x}_{1:M})^T + \mathbf{W} \end{aligned} \quad (1)$$

where $\mathbf{K}_n \in \mathbb{C}^{N_R \times N_T}$ (resp. $\mathbf{K}(\mathbf{x}_{1:M}, \boldsymbol{\tau})$) denotes the measured (resp. noise-free) MDM. Differently $\mathbf{W} \in \mathbb{C}^{N_R \times N_T}$ is a noise matrix s.t. $\text{vec}(\mathbf{W}) \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_N, \sigma_w^2 \mathbf{I}_N)$, where $N \triangleq N_T N_R$. Additionally, we have denoted: (i) the vector of scattering coefficients as $\boldsymbol{\tau} \triangleq [\tau_1 \ \cdots \ \tau_M]^T \in \mathbb{C}^{M \times 1}$; (ii) (b) the Tx (resp. Rx) array matrix as $\mathbf{G}_t(\mathbf{x}_{1:M}) \in \mathbb{C}^{N_T \times M}$ (resp. $\mathbf{G}_r(\mathbf{x}_{1:M}) \in \mathbb{C}^{N_R \times M}$), whose (i, j) th entry equals $\mathcal{G}(\tilde{\mathbf{r}}_i, \mathbf{x}_j)$ (resp. $\mathcal{G}(\bar{\mathbf{r}}_j, \mathbf{x}_i)$), where $\mathcal{G}(\cdot, \cdot)$ denotes the (scalar) background *Green function* [7]. Also, j th column $\mathbf{g}_t(\mathbf{x}_j)$ (resp. $\mathbf{g}_r(\mathbf{x}_j)$) of $\mathbf{G}_t(\mathbf{x}_{1:M})$ (resp. $\mathbf{G}_r(\mathbf{x}_{1:M})$) denotes the Tx (resp. Rx) Green's function vector evaluated at \mathbf{x}_j . In Eq. (2) the scattering matrix $\mathbf{M}(\mathbf{x}_{1:M}, \boldsymbol{\tau}) \in \mathbb{C}^{M \times M}$ equals $\mathbf{M}(\mathbf{x}_{1:M}, \boldsymbol{\tau}) \triangleq \text{diag}(\boldsymbol{\tau})$ for BA model [7], while $\mathbf{M}(\mathbf{x}_{1:M}, \boldsymbol{\tau}) \triangleq [\text{diag}^{-1}(\boldsymbol{\tau}) - \mathbf{S}(\mathbf{x}_{1:M})]^{-1}$ in the case of FL model [22], where the (m, n) th entry of $\mathbf{S}(\mathbf{x}_{1:M})$ equals $\mathcal{G}(\mathbf{x}_m, \mathbf{x}_n)$ when $m \neq n$ and zero otherwise. We recall that our null-spectrum analysis of TR-MUSIC is *general* and can be applied to both scattering models.

Finally, we define the SNR $\triangleq \|\mathbf{K}(\mathbf{x}_{1:M}, \boldsymbol{\tau})\|_F^2 / (\sigma_w^2 N_T N_R)$ and, for notational convenience, $N_{\text{Rdof}} \triangleq (N_R - M)$ and $N_{\text{Tdof}} \triangleq (N_T - M)$ as the dimensions of the left and right orthogonal subspaces, whereas $N_{\text{dof}} \triangleq (N_{\text{Rdof}} + N_{\text{Tdof}})$.

A. TR-MUSIC Spatial Spectrum

Several TR-MUSIC variants have been proposed in the literature for the non co-located setup [8]. A first approach consists in using the so-called *Rx mode TR-MUSIC*, which evaluates the *null (or spatial) spectrum* (assuming $M < N_R$):

$$\mathcal{P}_r(\mathbf{x}; \tilde{\mathbf{U}}_n) \triangleq \bar{\mathbf{g}}_r(\mathbf{x})^\dagger \tilde{\mathbf{P}}_{r,n} \bar{\mathbf{g}}_r(\mathbf{x}) = \left\| \tilde{\mathbf{U}}_n^\dagger \bar{\mathbf{g}}_r(\mathbf{x}) \right\|^2, \quad (3)$$

where $\tilde{\mathbf{U}}_n \in \mathbb{C}^{N_R \times N_{\text{Rdof}}}$ is the matrix of left singular vectors of \mathbf{K}_n spanning the noise subspace, $\bar{\mathbf{g}}_r(\mathbf{x}) \triangleq \mathbf{g}_r(\mathbf{x}) / \|\mathbf{g}_r(\mathbf{x})\|$ is the unit-norm Rx Green vector function and $\tilde{\mathbf{P}}_{r,n} \triangleq (\tilde{\mathbf{U}}_n \tilde{\mathbf{U}}_n^\dagger)$ (i.e. the “noisy” projector into the left noise subspace). A dual approach, denoted as *Tx mode TR-MUSIC*, constructs the null spectrum (assuming $M < N_T$):

$$\mathcal{P}_t(\mathbf{x}; \tilde{\mathbf{V}}_n) \triangleq \bar{\mathbf{g}}_t(\mathbf{x})^T \tilde{\mathbf{P}}_{t,n} \bar{\mathbf{g}}_t(\mathbf{x})^* = \left\| \tilde{\mathbf{V}}_n^\dagger \bar{\mathbf{g}}_t^*(\mathbf{x}) \right\|^2, \quad (4)$$

where $\tilde{\mathbf{V}}_n \in \mathbb{C}^{N_T \times N_{\text{Tdof}}}$ is the matrix of right singular vectors of \mathbf{K}_n spanning the noise subspace, $\bar{\mathbf{g}}_t(\mathbf{x}) \triangleq \mathbf{g}_t(\mathbf{x}) / \|\mathbf{g}_t(\mathbf{x})\|$ is the unit-norm Tx Green vector function and $\tilde{\mathbf{P}}_{t,n} \triangleq (\tilde{\mathbf{V}}_n \tilde{\mathbf{V}}_n^\dagger)$ (i.e. the “noisy” projector into the right noise subspace). Finally, a combined version of two modes, named *generalized TR-MUSIC*, is built as (assuming $M < \min\{N_T, N_R\}$) [8]:

$$\mathcal{P}_{\text{tr}}(\mathbf{x}; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n) \triangleq \mathcal{P}_t(\mathbf{x}; \tilde{\mathbf{V}}_n) + \mathcal{P}_r(\mathbf{x}; \tilde{\mathbf{U}}_n). \quad (5)$$

Usually, the M largest local maxima of $\mathcal{P}_r(\mathbf{x}; \tilde{\mathbf{U}}_n)^{-1}$, $\mathcal{P}_t(\mathbf{x}; \tilde{\mathbf{V}}_n)^{-1}$ and $\mathcal{P}_{\text{tr}}(\mathbf{x}; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n)^{-1}$ are chosen as the estimates $\{\hat{\mathbf{x}}_k\}_{k=1}^M$. Indeed, it can be shown that Eq. (3) (resp. Eq. (4)) equals zero when \mathbf{x} equals one among $\{\mathbf{x}_k\}_{k=1}^M$ in the noise-free case, since when $\tilde{\mathbf{U}}_n = \mathbf{U}_n$ (resp. $\tilde{\mathbf{V}}_n = \mathbf{V}_n$) this reduces to the eigenvector matrix spanning the left (resp. right) noise subspace of $\mathbf{K}(\mathbf{x}_{1:M}, \boldsymbol{\tau})$ [7]. Similar conclusions hold for $\mathcal{P}_{\text{tr}}(\mathbf{x}; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n)$ in a noise-free condition.

B. Review of Results on SVD Perturbation

We consider a rank deficient matrix $\mathbf{A} \in \mathbb{C}^{R \times T}$ with rank $\delta < \min\{R, T\}$, whose SVD $\mathbf{A} = \mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^\dagger$ is rewritten as:

$$\mathbf{A} = \begin{pmatrix} \mathbf{U}_s & \mathbf{U}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_s & \mathbf{0}_{\delta \times \bar{\delta}} \\ \mathbf{0}_{\bar{\delta} \times \delta} & \mathbf{0}_{\bar{\delta} \times \bar{\delta}} \end{pmatrix} \begin{pmatrix} \mathbf{V}_s^\dagger \\ \mathbf{V}_n^\dagger \end{pmatrix}, \quad (6)$$

where $\bar{\delta} \triangleq (R - \delta)$ and $\bar{\delta} \triangleq (T - \delta)$, respectively. Also, $\mathbf{U}_s \in \mathbb{C}^{R \times \delta}$ and $\mathbf{V}_s \in \mathbb{C}^{T \times \delta}$ (resp. $\mathbf{U}_n \in \mathbb{C}^{R \times \bar{\delta}}$ and $\mathbf{V}_n \in \mathbb{C}^{T \times \bar{\delta}}$) denote the left and right singular vectors of signal (resp. orthogonal) subspaces in Eq. (6), while $\boldsymbol{\Sigma}_s \in \mathbb{R}^{\delta \times \delta}$ collects the (> 0) singular values of the signal subspace. Then, consider $\tilde{\mathbf{A}} = (\mathbf{A} + \mathbf{N})$, where \mathbf{N} is a perturbing term. Similarly to (6), the SVD $\tilde{\mathbf{A}} = \tilde{\mathbf{U}} \tilde{\boldsymbol{\Sigma}} \tilde{\mathbf{V}}^\dagger$ is rewritten as

$$\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{\mathbf{U}}_s & \tilde{\mathbf{U}}_n \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\Sigma}}_s & \mathbf{0}_{\delta \times \bar{\delta}} \\ \mathbf{0}_{\bar{\delta} \times \delta} & \tilde{\boldsymbol{\Sigma}}_n \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{V}}_s^\dagger \\ \tilde{\mathbf{V}}_n^\dagger \end{pmatrix}, \quad (7)$$

showing the effect of \mathbf{N} on the spectral representation⁴ of $\tilde{\mathbf{A}}$, highlighting the change of the left and right principal directions. We are here concerned with the perturbations pertaining to $\tilde{\mathbf{U}}_n$ and $\tilde{\mathbf{V}}_n$, stressed as $\tilde{\mathbf{U}}_n = \mathbf{U}_n + \Delta \mathbf{U}_n$ and $\tilde{\mathbf{V}}_n = \mathbf{V}_n + \Delta \mathbf{V}_n$, where $\Delta(\cdot)$ terms are generally complicated functions of \mathbf{N} . However, when \mathbf{N} has a “small magnitude” compared to \mathbf{A} (see [31]), a 1st-order perturbation (i.e. $\Delta(\cdot)$ are approximated as linear with \mathbf{N}), will be accurate [24]. The key result is that perturbed orthogonal left subspace $\tilde{\mathbf{U}}_n$ (resp. right subspace $\tilde{\mathbf{V}}_n$) is spanned by $\mathbf{U}_n + \mathbf{U}_s \mathbf{B}$ (resp. $\mathbf{V}_n + \mathbf{V}_s \bar{\mathbf{B}}$), where norm (any sub-multiplicative one, such as ℓ_2 or $\|\cdot\|_F$ norm) of \mathbf{B} (resp. $\bar{\mathbf{B}}$) is of the same order of that of

³The number of scatterers M is assumed to be known, as usually done in array-processing literature [29].

⁴Indeed, as opposed to Eq. (6), $\tilde{\mathbf{A}}$ may be full-rank in general.

N . Intuitively, a *small perturbation* is observed at *high-SNR*. The expressions for $\Delta \mathbf{U}_n$ and $\Delta \mathbf{V}_n$, at 1st-order, are⁵ [32]:

$$\Delta \mathbf{U}_n = -(\mathbf{A}^-)^\dagger \mathbf{N}^\dagger \mathbf{U}_n; \quad \Delta \mathbf{V}_n = -(\mathbf{A}^-) \mathbf{N} \mathbf{V}_n; \quad (8)$$

where we have exploited $\mathbf{A}^- = \mathbf{V}_s \Sigma_s^{-1} \mathbf{U}_s^\dagger$ [33].

III. NULL-SPECTRUM ANALYSIS

First, we observe that the null spectrums at scatterer positions $\mathcal{P}_r(\mathbf{x}_k; \tilde{\mathbf{U}}_n)$, $\mathcal{P}_t(\mathbf{x}_k; \tilde{\mathbf{V}}_n)$ and $\mathcal{P}_{tr}(\mathbf{x}_k; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n)$, $k \in \{1, \dots, M\}$, in Eqs. (3), (4) and (5) can be simplified, using $\tilde{\mathbf{U}}_n = \mathbf{U}_n + \Delta \mathbf{U}_n$ and $\tilde{\mathbf{V}}_n = \mathbf{V}_n + \Delta \mathbf{V}_n$ and exploiting the properties⁶ $\mathbf{U}_n^\dagger \tilde{\mathbf{g}}_r(\mathbf{x}_k) = \mathbf{0}_{N_{\text{Rdof}}}$ and $\mathbf{V}_n^\dagger \tilde{\mathbf{g}}_t^*(\mathbf{x}_k) = \mathbf{0}_{N_{\text{Tdof}}}$, as

$$\mathcal{P}_r(\mathbf{x}_k; \tilde{\mathbf{U}}_n) = \|\xi_{r,k}\|^2, \quad \mathcal{P}_t(\mathbf{x}_k; \tilde{\mathbf{V}}_n) = \|\xi_{t,k}\|^2, \quad (9)$$

where $\xi_{r,k} \triangleq \Delta \mathbf{U}_n^\dagger \tilde{\mathbf{g}}_r(\mathbf{x}_k) \in \mathbb{C}^{N_{\text{Rdof}} \times 1}$ and $\xi_{t,k} \triangleq \Delta \mathbf{V}_n^\dagger \tilde{\mathbf{g}}_t^*(\mathbf{x}_k) \in \mathbb{C}^{N_{\text{Tdof}} \times 1}$, respectively. Similarly,

$$\mathcal{P}_{tr}(\mathbf{x}_k; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n) = \|\xi_{t,k}\|^2 + \|\xi_{r,k}\|^2 = \|\xi_k\|^2, \quad (10)$$

where $\xi_k \triangleq [\xi_{r,k}^T \ \xi_{t,k}^T]^T \in \mathbb{C}^{N_{\text{dof}} \times 1}$. Thus, to characterize $\mathcal{P}_r(\mathbf{x}_k; \tilde{\mathbf{U}}_n)$, $\mathcal{P}_t(\mathbf{x}_k; \tilde{\mathbf{V}}_n)$ and $\mathcal{P}_{tr}(\mathbf{x}_k; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n)$, it suffices to study the random vector ξ_k . Indeed, the marginal pdfs of $\xi_{r,k}$ and $\xi_{t,k}$ are easily drawn from that of ξ_k . As a byproduct, ξ_k definition also allows an elegant and simpler MSE analysis with respect to [21], as it can be shown that the position-error of the estimates with Tx mode ($\Delta \mathbf{x}_{T,k}$), Rx mode ($\Delta \mathbf{x}_{R,k}$) and generalized ($\Delta \mathbf{x}_{TR,k}$) TR-MUSIC can be expressed as $\Delta \mathbf{x}_{T,k} \approx -\Gamma_{T,k}^{-1} \Re\{J_{T,k}^T \mathbf{V}_n \xi_{t,k}\}$, $\Delta \mathbf{x}_{R,k} \approx -\Gamma_{R,k}^{-1} \Re\{J_{R,k}^T \mathbf{U}_n \xi_{r,k}\}$ and $\Delta \mathbf{x}_{TR,k} \approx -\Gamma_{TR,k}^{-1} \Re\{[(J_{R,k}^\dagger \mathbf{U}_n) \ (J_{T,k}^\dagger \mathbf{V}_n)] \xi_k\}$, respectively, where $J_{T,k}$, $J_{R,k}$, $\Gamma_{T,k}$, $\Gamma_{R,k}$ and $\Gamma_{TR,k}$ are suitably defined *known* matrices (see [21]). Clearly, finding the exact pdf of ξ_k is hard, as $\Delta \mathbf{U}_n$ and $\Delta \mathbf{V}_n$ are generally complicated functions of the unknown *perturbing* matrix \mathbf{W} .

However, $\Delta \mathbf{U}_n$ and $\Delta \mathbf{V}_n$ assume a (tractable) closed form with a 1st-order approximation (see Eq. (8)). This approximation holds tightly at high-SNR, as \mathbf{W} will be statistically “small” compared to noise-free MDM $\mathbf{K}(\mathbf{x}_{1:M}, \boldsymbol{\tau})$. Hence, at high-SNR, ξ_k is (approximately) expressed in terms of \mathbf{W} as:

$$\xi_k = \begin{bmatrix} \xi_{r,k} \\ \xi_{t,k} \end{bmatrix} \approx \begin{bmatrix} -\mathbf{U}_n^\dagger \mathbf{W} \mathbf{t}_{r,k} \\ -\mathbf{V}_n^\dagger \mathbf{W}^\dagger \mathbf{t}_{t,k} \end{bmatrix}, \quad (11)$$

where $\mathbf{t}_{r,k} \triangleq \mathbf{K}^-(\mathbf{x}_{1:M}, \boldsymbol{\tau}) \tilde{\mathbf{g}}_r(\mathbf{x}_k) \in \mathbb{C}^{N_T \times 1}$ and $\mathbf{t}_{t,k} \triangleq \mathbf{K}^-(\mathbf{x}_{1:M}, \boldsymbol{\tau})^\dagger \tilde{\mathbf{g}}_t^*(\mathbf{x}_k) \in \mathbb{C}^{N_R \times 1}$ are *deterministic*. Since the vector ξ_k is linear⁷ with the noise matrix \mathbf{W} , it will be Gaussian distributed; thus we only need to evaluate its moments up to the 2nd order to characterize it completely. Hereinafter we only sketch the main steps and provide the detailed proof as supplementary material. First, the mean vector

⁵We notice that in obtaining Eq. (8), “in-space” perturbations (e.g. the contribution to $\Delta \mathbf{U}_n$ depending on \mathbf{U}_n) are not considered, though they have been shown to be linear with N (and thus *not negligible* at first-order) [32]. The reason is that these terms do not affect performance analysis of TR-MUSIC null-spectrum when evaluated at scatterers positions $\{\mathbf{x}_k\}_{k=1}^M$, due to the null spectrum orthogonality property.

⁶Such conditions directly follow from orthogonality between left (resp. right) signal and orthogonal subspaces \mathbf{U}_s and \mathbf{U}_n (resp. \mathbf{V}_s and \mathbf{V}_n).

⁷In the following of the letter we will implicitly mean that the results hold “approximately” in the high-SNR regime.

$\mathbb{E}\{\xi_{r,k}^T \ \xi_{t,k}^T\} = \mathbf{0}_{N_{\text{dof}}}$, exploiting $\mathbb{E}\{\mathbf{W}\} = \mathbf{0}_{N_R \times N_T}$. Secondly, the *covariance matrix* $\Xi_k \triangleq \mathbb{E}\{\xi_k \xi_k^\dagger\}$ (since $\mathbb{E}\{\xi_k\} = \mathbf{0}_{N_{\text{dof}}}$) is given in closed-form as:

$$\Xi_k = \begin{bmatrix} \sigma_w^2 \|\mathbf{t}_{r,k}\|^2 \mathbf{I}_{N_{\text{Rdof}}} & \mathbf{0}_{N_{\text{Rdof}} \times N_{\text{Tdof}}} \\ \mathbf{0}_{N_{\text{Tdof}} \times N_{\text{Rdof}}} & \sigma_w^2 \|\mathbf{t}_{t,k}\|^2 \mathbf{I}_{N_{\text{Tdof}}} \end{bmatrix}. \quad (12)$$

The above result is based on circularity of the entries of \mathbf{W} , along with their mutual independence. Thirdly, aiming at completing the statistical characterization, we evaluate the *pseudo-covariance matrix* $\Psi_k \triangleq \mathbb{E}\{\xi_k \xi_k^T\}$ (since $\mathbb{E}\{\xi_k\} = \mathbf{0}_{N_{\text{dof}}}$), whose closed-form is $\Psi_k = \mathbf{0}_{N_{\text{dof}} \times N_{\text{dof}}}$. The latter result is based on circularity of the entries of \mathbf{W} , along with their mutual independence and exploiting the results $\mathbf{V}_n^\dagger \mathbf{t}_{r,k} = \mathbf{0}_{N_{\text{Tdof}}}$ and $\mathbf{U}_n^\dagger \mathbf{t}_{t,k} = \mathbf{0}_{N_{\text{Rdof}}}$, arising from subspaces orthogonality $\mathbf{V}_n^\dagger \mathbf{V}_s = \mathbf{0}_{N_{\text{Tdof}} \times M}$ and $\mathbf{U}_n^\dagger \mathbf{U}_s = \mathbf{0}_{N_{\text{Rdof}} \times M}$.

Therefore, in summary $\xi_k \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{N_{\text{dof}}}, \Xi_k)$, i.e. a *proper* complex Gaussian vector [34]. Similarly, it is readily inferred that $\xi_{r,k} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{N_{\text{Rdof}}}, \sigma_w^2 \|\mathbf{t}_{r,k}\|^2 \mathbf{I}_{N_{\text{Rdof}}})$ and $\xi_{t,k} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{N_{\text{Tdof}}}, \sigma_w^2 \|\mathbf{t}_{t,k}\|^2 \mathbf{I}_{N_{\text{Tdof}}})$, respectively, i.e. they are *independent proper* Gaussian vectors. Clearly, since $\xi_{r,k}$ and $\xi_{t,k}$ have zero mean and scaled-identity covariance, the corresponding *variance-normalized* energies $\|\xi_{r,k}\|^2 / (\sigma_w^2 \|\mathbf{t}_{r,k}\|^2) \sim \mathcal{C}\chi_{N_{\text{Rdof}}}^2$ and $\|\xi_{t,k}\|^2 / (\sigma_w^2 \|\mathbf{t}_{t,k}\|^2) \sim \mathcal{C}\chi_{N_{\text{Tdof}}}^2$, respectively (i.e. they are chi-square distributed). Interestingly these DOFs coincide with those available for TR-MUSIC localization through Rx and Tx modes, respectively.

Based on these considerations, the means of the null-spectrum for Tx and Rx modes are $\mathbb{E}\{\|\xi_{r,k}\|^2\} = \sigma_w^2 \|\mathbf{t}_{r,k}\|^2 N_{\text{Rdof}}$ and $\mathbb{E}\{\|\xi_{t,k}\|^2\} = \sigma_w^2 \|\mathbf{t}_{t,k}\|^2 N_{\text{Tdof}}$, respectively, whereas for generalized null-spectrum $\mathbb{E}\{\|\xi_k\|^2\} = \mathbb{E}\{\|\xi_{r,k}\|^2\} + \mathbb{E}\{\|\xi_{t,k}\|^2\}$ (by linearity). By similar reasoning, the variances for Tx and Rx modes are given by $\text{var}\{\|\xi_{r,k}\|^2\} = \sigma_w^4 \|\mathbf{t}_{r,k}\|^4 N_{\text{Rdof}}$ and $\text{var}\{\|\xi_{t,k}\|^2\} = \sigma_w^4 \|\mathbf{t}_{t,k}\|^4 N_{\text{Tdof}}$, respectively, whereas for the generalized null-spectrum $\text{var}\{\|\xi_k\|^2\} = \text{var}\{\|\xi_{r,k}\|^2\} + \text{var}\{\|\xi_{t,k}\|^2\}$ (by independence of $\xi_{r,k}$ and $\xi_{t,k}$).

Hence, once we have obtained the mean and the variance of $\mathcal{P}_r(\mathbf{x}_k; \tilde{\mathbf{U}}_n)$, $\mathcal{P}_t(\mathbf{x}_k; \tilde{\mathbf{V}}_n)$ and $\mathcal{P}_{tr}(\mathbf{x}_k; \tilde{\mathbf{U}}_n, \tilde{\mathbf{V}}_n)$, respectively, we can consider the *Normalized Standard Deviation* (NSD), generically defined as

$$\text{NSD}_k \triangleq \sqrt{\text{var}\{\mathcal{P}(\mathbf{x}_k; \cdot)\} / \mathbb{E}\{\mathcal{P}(\mathbf{x}_k; \cdot)\}}. \quad (13)$$

Clearly, the lower the NSD, the higher the null-spectrum stability at \mathbf{x}_k [23]. For Rx and Tx modes it follows that $\text{NSD}_{r,k} = 1/\sqrt{N_{\text{Rdof}}}$ and $\text{NSD}_{t,k} = 1/\sqrt{N_{\text{Tdof}}}$, respectively. It is apparent that in both cases the NSD *does not depend* (at high SNR) on the scatterers and measurement setup, as well as σ_w^2 , but only on the (complex) DOFs, being equal to N_{Rdof} and N_{Tdof} , respectively. Thus, the NSD becomes (asymptotically) small only when the number of scatterers is few compared to the Tx (resp. Rx) elements of the array. Those results are analogous to the case of MUSIC null-spectrum for DOA, whose NSD depends on the DOFs, namely the difference between the (Rx) array size and the number of sources [23]. Differently, the NSD for generalized null spectrum equals

$$\text{NSD}_k = \frac{\sqrt{\|\mathbf{t}_{r,k}\|^4 N_{\text{Rdof}} + \|\mathbf{t}_{t,k}\|^4 N_{\text{Tdof}}}}{\|\mathbf{t}_{r,k}\|^2 N_{\text{Rdof}} + \|\mathbf{t}_{t,k}\|^2 N_{\text{Tdof}}}. \quad (14)$$

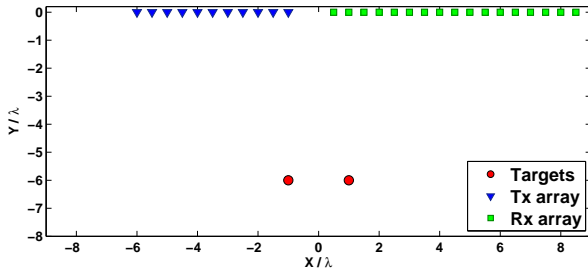


Figure 1. Geometry for the considered imaging problem in 2-D space.

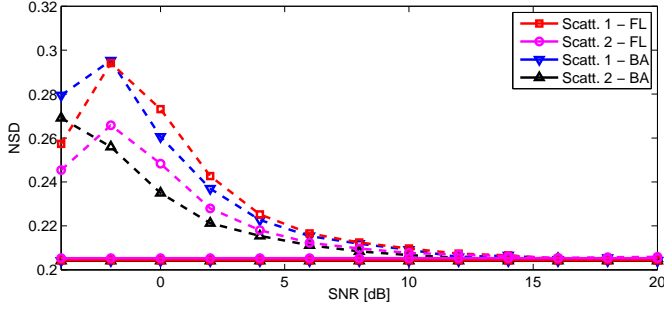


Figure 2. NSD (generalized null spectrum) vs. SNR; theoretical (Eq. (14), solid lines) vs. MC-based (dashed lines) performance.

Eq. (14) underlines (i) a clear dependence of generalized null-spectrum NSD on scatterers and measurement setup and (ii) independence from the noise level σ_w^2 . Also, it is apparent that when $\|\mathbf{t}_{r,k}\| \approx 0$ (resp. $\|\mathbf{t}_{t,k}\| \approx 0$) the expression reduces to $\text{NSD}_k \approx 1/\sqrt{N_{\text{Tdof}}}$ (resp. $\text{NSD}_k \approx 1/\sqrt{N_{\text{Rdof}}}$), i.e. the NSD is *dominated* by Tx (resp. Rx) mode stability. Finally, the same equation is exploited to obtain the conditions ensuring that generalized spectrum is “more stable” than Tx and Rx modes ($\text{NSD}_k \leq \text{NSD}_{t,k}$ and $\text{NSD}_k \leq \text{NSD}_{r,k}$, respectively), expressed as the pair of inequalities

$$\begin{cases} \frac{1}{2} [1 - N_{\text{Rdof}}/N_{\text{Tdof}}] \leq (\|\mathbf{t}_{t,k}\| / \|\mathbf{t}_{r,k}\|)^2 & (\text{Tx}) \\ \frac{1}{2} [1 - N_{\text{Tdof}}/N_{\text{Rdof}}] \leq (\|\mathbf{t}_{r,k}\| / \|\mathbf{t}_{t,k}\|)^2 & (\text{Rx}) \end{cases} \quad (15)$$

Clearly, when $N_R > N_T$ (resp. $N_T > N_R$) the inequality regarding the Tx (resp. Rx) mode is always verified as the left-hand side is always negative. Also, in the special case $N_T = N_R$ the left-hand side is always zero for both inequalities.

IV. NUMERICAL RESULTS

In this section we confirm our findings through simulations, focusing on 2-D localization, with Green function⁸ being $\mathcal{G}(\mathbf{x}', \mathbf{x}) = H_0^{(1)}(\kappa \|\mathbf{x}' - \mathbf{x}\|)$. Here $H_n^{(1)}(\cdot)$ and $\kappa = 2\pi/\lambda$ denote the n th order *Hankel* function of the 1st kind and the wavenumber (λ is the wavelength), respectively. First, we consider a setup with $\lambda/2$ -spaced Tx/Rx arrays ($N_T = 11$ and $N_R = 17$, respectively, see Fig. 1). Secondly, to quantify the level of multiple scattering (as in [8]) we define the index $\eta \triangleq \|\mathbf{K}_f(\mathbf{x}_{1:M}, \boldsymbol{\tau}) - \mathbf{K}_b(\mathbf{x}_{1:M}, \boldsymbol{\tau})\|_F / \|\mathbf{K}_b(\mathbf{x}_{1:M}, \boldsymbol{\tau})\|_F$, where $\mathbf{K}_b(\mathbf{x}_{1:M}, \boldsymbol{\tau})$ and $\mathbf{K}_f(\mathbf{x}_{1:M}, \boldsymbol{\tau})$ denote the MDMs generated via BA and FL models, respectively. Finally, for simplicity we

⁸We discard the irrelevant constant term $j/4$.

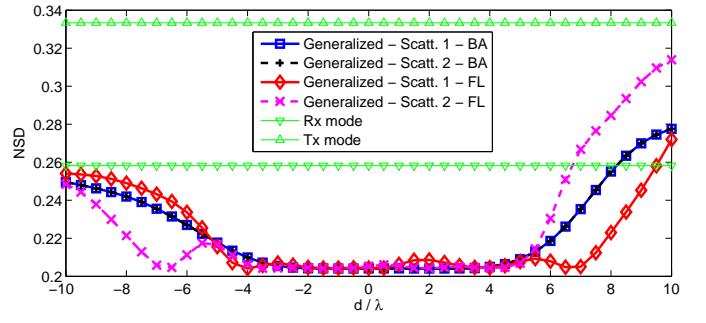


Figure 3. Theoretical NSD vs. scatterers rigid shift d ; two targets located at $(\mathbf{x}_1/\lambda) = [(-1-d) \ -6]^T$ and $(\mathbf{x}_2/\lambda) = [(1-d) \ -6]^T$.

consider $M = 2$ targets located at $(\mathbf{x}_1/\lambda) = [-1 \ -6]^T$ and $(\mathbf{x}_2/\lambda) = [+1 \ -6]^T$ and having scattering coefficients $\boldsymbol{\tau} = [3 \ 4]^T$; thus $\eta = (0.7445)$.

Then, we compare the asymptotic NSD (Eq. (14), solid lines) with the true ones obtained via Monte Carlo (MC) simulation (dashed lines, 10^5 runs), focusing only on the generalized null-spectrum for brevity. To this end, Fig. 2 depicts the null-spectrum NSD vs. SNR for the two targets being considered, both for FL and BA models. It is apparent that, as the SNR increases, the theoretical results tightly approximate the MC-based ones, with approximations deemed accurate above $\text{SNR} \approx 10$ dB. Differently, in Fig. 3, we plot the asymptotic NSD of the three TR-MUSIC variants vs. d , where $(\mathbf{x}_1/\lambda) = [(-1-d) \ -6]^T$ and $(\mathbf{x}_2/\lambda) = [(1-d) \ -6]^T$ (i.e. a rigid shift of the two scatterers), in order to investigate the potentially improved asymptotic stability (viz. NSD) of the generalized spectrum in comparison to Tx and Rx modes. It is apparent that the gain is significant when $d \in (-5, 5)$, while outside this interval the NSD expression is either dominated by Tx or Rx mode, which for the present case $\text{NSD}_{t,k} = 1/\sqrt{11-2} \approx 0.33$ and $\text{NSD}_{r,k} = 1/\sqrt{17-2} \approx 0.26$, with the generalized NSD never above that of $\text{NSD}_{t,k}$ (as dictated from Eq. (15)).

V. CONCLUSIONS

We provided an asymptotic (high-SNR) analysis of TR-MUSIC null-spectrum in a non-colocated multistatic setup, by taking advantage of the 1st-order perturbation of the SVD of the MDM. Three different variants of TR-MUSIC were analyzed (i.e. Tx mode, Rx mode and generalized), based on the characterization of a certain complex-valued Gaussian vector. This allowed to obtain the asymptotic NSD (a measure of null-spectrum stability) for all the three imaging procedures. While similar results as the DOA setup were obtained for Tx and Rx modes, it was shown a clear dependence of generalized null-spectrum NSD on the scatterer and measurement setup. Finally, its potential stability advantage was investigated in comparison to Tx and Rx modes. Future works will analyze mutual coupling, antenna pattern and polarization effects [35], [36], and propagation in inhomogeneous (random) media [37].

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